

## 2-D Emittance Equation with Acceleration and Compression

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Since both acceleration and compression are required for an Inertial Fusion Driver, the understanding of their effect on the beam quality, emittance, is important. This report attempts to generalize the usual emittance formula<sup>1,2</sup> for the drifting beam to include these effects. The derivation of the 2-D emittance equation is carried out and a comparison with the particle code results is given. The 2-D emittance at a given axial location is reasonable to consider for a long beam, particularly with velocity tilt; transverse emittance averaged over the entire bunch is not a useful quantity.

### 1. Derivation of the emittance equation.

The coordinates system used is (t,x,y) rather than the usual (s,x,y), since the transverse momentum is invariant under longitudinal acceleration. The following moment equations can be derived from the Vlasov equation:

$$\frac{dn}{dt} + \nabla n \mathbf{v} = \frac{n}{\lambda} \quad (1.a)$$

$$\frac{d\langle x^2 \rangle}{dt} - 2 \langle x \frac{p_x}{m} \rangle = 0 \quad (1.b)$$

$$\frac{d\langle x p_x \rangle}{dt} - \frac{p_x^2}{m} + k_x \langle x^2 \rangle - \frac{e}{2} \langle x E_x \rangle = 0 \quad (1.c)$$

$$\frac{d\langle p_x^2 \rangle}{dt} + 2 k_x \langle x p_x \rangle - \frac{2e}{2} \langle p_x E_x \rangle = 0 \quad (1.d)$$

$$= N / \frac{dN}{dt}$$

$$N = \int n \, dx \, dy$$

where  $\lambda$  is the inverse of the axial compression rate and N is the line number density. Also,  $k_x$  is the focusing strength of the quadrupole in x-direction and m is the rest mass.

The definition of normalized emittance and its time derivative are given as follows:

$$\frac{2}{x} = \frac{1}{m^2 c^2} [ \langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 ] \quad (2.a)$$

$$\frac{d}{dt} \frac{2}{x} = \frac{2e}{2m^2 c^2} [ \langle x^2 \rangle \langle p_x E_x \rangle - \langle x p_x \rangle \langle x E_x \rangle ] \quad (2.b)$$

After dividing by  $\langle x^2 \rangle$  and adding the corresponding y equation, it becomes:

$$\frac{1}{\langle x^2 \rangle} \frac{d}{dt} \frac{2}{x} + \frac{1}{\langle y^2 \rangle} \frac{d}{dt} \frac{2}{y} = \frac{2e}{2m^2} [ \langle \mathbf{p} \cdot \mathbf{E} \rangle - I ] \quad (3)$$

where 
$$I = \frac{\langle x p_x \rangle}{\langle x^2 \rangle} \langle x E_x \rangle + \frac{\langle y p_y \rangle}{\langle y^2 \rangle} \langle y E_y \rangle.$$

The  $\langle \mathbf{p} \cdot \mathbf{E} \rangle$  term is the work done on the particles and it can be rewritten as:

$$\frac{e}{m} \langle \mathbf{p} \cdot \mathbf{E} \rangle = \frac{\mathbf{j} \cdot \mathbf{E} dV}{N} = \frac{1}{N} \nabla \cdot \mathbf{j} dV = - \frac{1}{N} \left[ \frac{dW}{dt} - \frac{2W}{dt} \right] \quad (4)$$

where  $\mathbf{E} = -\nabla V$  and the continuity equation has been used.  $W$  is the electrostatic energy per unit length and  $\frac{2W}{dt}$  is the convective energy flow due to axial compression.

The second term on the right hand side of the emittance equation is also expressed as:

$$\begin{aligned} \frac{e}{m} I &= \frac{e}{2} \left[ \frac{\left( \frac{d\langle x^2 \rangle}{dt} \right)}{\langle x^2 \rangle} \langle x E_x \rangle + \frac{\left( \frac{d\langle y^2 \rangle}{dt} \right)}{\langle y^2 \rangle} \langle y E_y \rangle \right] \\ &= \frac{1}{N} \left[ \frac{dW_u}{dt} - \frac{2W_u}{dt} \right] \end{aligned} \quad (5)$$

where 
$$W_u = W_0 \left[ 1 + 4 \ln \frac{2R_c}{a+b} \right]$$

$$W_0 = \frac{N^2 e^2}{4}.$$

Here  $R_c$  is the radius of the outside conductor and  $R_c$  is assumed to be large enough to ignore detailed shape of the boundary. Also,  $W_u$  is the electrostatic energy per unit length for the uniform density profile beam with the same  $\langle x^2 \rangle$  and  $\langle y^2 \rangle$ . The relation is strictly

true only for a charge distribution with elliptic symmetry, i.e.,  $n = n \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right)$  at all time;<sup>3</sup> however numerical simulations show it is not a strong requirement.<sup>2</sup>

Equation (3) becomes:

$$\begin{aligned} \frac{1}{\langle x^2 \rangle} \frac{d}{dt} \frac{2}{x} + \frac{1}{\langle y^2 \rangle} \frac{d}{dt} \frac{2}{y} &= - \frac{2}{mc^2} \frac{1}{N} \left[ \frac{d}{dt} (W - W_u) - \frac{2}{N} (W - W_u) \right] \\ &= - \frac{Q}{2} \frac{d}{dt} \left[ \frac{W - W_u}{W_0} \right] \end{aligned} \quad (6)$$

where  $Q = \frac{Ne^2}{mc^2}$ .

If we assume  $\frac{2}{x} = \frac{2}{y} = \frac{2}{n}$ , it can be simplified further to the following form.\*

$$\begin{aligned} \frac{d}{dt} \frac{2}{n} &= - \frac{Q}{2} \frac{d}{dt} \left[ \frac{W - W_u}{W_0} \right] / \left( \frac{1}{\langle x^2 \rangle} + \frac{1}{\langle y^2 \rangle} \right) \\ &= - \frac{Q \langle r^2 \rangle (1 - e^2)}{8} \frac{d}{dt} \left[ \frac{W - W_u}{W_0} \right] \end{aligned} \quad (7)$$

where  $\langle r^2 \rangle = \langle x^2 + y^2 \rangle$  and  $e = \frac{\langle x^2 \rangle - \langle y^2 \rangle}{\langle r^2 \rangle}$  which becomes zero for a circular cross section. This can be trivially recast in terms of  $s$ . (Also it can be derived directly using an  $(s, x, y)$  coordinate system.)

$$\frac{d}{ds} \frac{2}{n} = - \frac{Q \langle r^2 \rangle (1 - e^2)}{8} \frac{d}{ds} \left[ \frac{W - W_u}{W_0} \right] \quad (8)$$

The rate of normalized emittance change is proportional to the rate of change of the normalized non-linear electrostatic energy, as for the drifting beam. The proportionality multiplier depends mainly on  $N$  and  $\langle r^2 \rangle$ , neglecting small changes of  $e$ . For a matched beam,  $e^2 < 10\%$  and will be neglected hereafter.

## 2. Applications

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\* The effect of small differences of emittance in  $x$  and  $y$  due to the FODO lattice structure becomes negligible, when the time average over a lattice period is taken.

Since the typical time scale for electrostatic energy changes is the plasma frequency, on the order of  $(\frac{0V_z}{L})$ , in contrast to the relatively slow rate of change of  $Q\langle r^2 \rangle$  for the adiabatically evolving beam, the right hand side can be approximated by the total derivative yielding a simple estimation formula for the final emittance. However, care is necessary for the case of slow driven electrostatic energy changes, as discussed later.

For the space charge dominated drifting beam with matched envelope, the emittance formula gives the following estimation as in the reference 1,

$$\frac{r_f^2}{i^2} - 1 = \frac{1}{4} \left[ \left( \frac{0^2}{2} - 1 \right) \left( \frac{W - W_u}{W_0} \right) \right]_i \quad (9)$$

where the final state is assumed to be of uniform density, which is the equilibrium energy state in the limit of infinite tune depression. The difference in the change of for the accelerating beam is minor, slightly less for the same  $0$ , since  $\langle r^2 \rangle$  becomes smaller for a gentle acceleration.

The axial compression will enhance the change of the emittance by increasing  $N$ , the final emittance will be somewhat higher than the drifting beam. (By a factor of  $\sim [1 + \frac{2L}{0V_z}]$ ).

As the beam is axially expanding, as in the head and tail end of the bunch, the radius of the beam decreases and the temperature rises resulting in less tune depression. At the same time the Debye length increases and the beam profile becomes more centrally peaked. Thus the normalized electrostatic energy increases and the emittance decreases. In the limit of extreme decompression from the uniform density distribution, simple estimation of final emittance can be made as follows:

$$\begin{aligned} \frac{r_f^2}{i^2} - 1 &= \frac{1}{i^2} \frac{d}{ds} \left[ \frac{d}{ds} \frac{Q\langle r^2 \rangle}{8} \right] \left[ \frac{W - W_u}{W_0} \right] \\ &= -\frac{1}{2} \frac{1}{i^2} \left[ \frac{Q}{8} \right]_i \left[ \frac{\langle r^2 \rangle_i + \langle r^2 \rangle_f}{2} \right] \left[ \frac{W - W_u}{W_0} \right]_f \\ &= -\frac{1}{16} \left( \frac{0}{0} + 1 \right) \left( \frac{0^2}{2} - 1 \right)_i \left[ \frac{W - W_u}{W_0} \right]_f \end{aligned} \quad (10)$$

where the small time derivative of  $\langle r^2 \rangle$  is ignored and  $Q$  is assumed to decrease linearly in time. The final radius has been determined by the initial emittance. This small emittance decrease at the head and tail of the beam has been observed in MBE-4 experiments.<sup>4</sup>

### 3. Comparison with particle code simulation

The 2-D PIC code SHIFTXY has been modified to accommodate acceleration and compression.<sup>5</sup> Acceleration is added by function kicks in the middle of the drift section and linear increase of energy with distance is assumed. Linear compression or decompression in distance is also assumed. Hollow initial density distribution of  $n \sim r^2$  for accelerated or compressed beam is assumed, where  $r^2 = (\frac{x^2}{a^2} + \frac{y^2}{b^2})$  and a and b are the major and minor radii of the beam. An initial Gaussian velocity distribution is used.

Figure 1 shows the time history of the unnormalized emittance for the case where the initial velocity has been doubled and N quadrupled through the 30 down stream lattice periods. Also shown is  $\sigma_z$  of the drifting beam. The emittance increases and oscillates rapidly for the first undepressed betatron period and settles down to a more quiet state with residual oscillations.<sup>6</sup> The difference of final normalized emittance of the two is rather small due to weak compression during the period of rapid emittance change. The final emittance of the various cases are summarized in Table 1.

The effect of the axial expansion on  $\sigma_z$  is shown in Figure 2. Q is decreased linearly to 0.16 times the initial value. Initial tune depression ( $\frac{\sigma_z}{\sigma_{z0}}$ ) is 0.2. The final emittance shows 6.0 % decrease compared with analytic estimation of 6.6 % using equation (10) with assumed final Gaussian density distribution.

### 4. Conclusion

The derived emittance equation which includes the effects of acceleration and compression shows good agreement with particle simulation and it supersedes the previous emittance formula for the drifting beam.

### References

- [1] T.P. Wangler, K.R. Crandall, T.S. Mills, and M. Reiser, *IEEE Trans. Nucl. Sci.* **NS-32**, 2196 (1985).
- [2] I. Hofmann and J. Struckmeier, *Particle Accelerators* **21**, 69 (1987).
- [3] F.J. Sacherer, *IEEE Trans. Nucl. Sci.* **18**, 1105 (1971).

- [4] H. Meuth, private communication.  
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 [6] O. A. Anderson, *Particle Accelerators* **21**, 197 (1988).

Table 1. Summary of the particle code simulation. The numbers in the brackets are the analytic estimations. The initial  $\theta_0$  is  $60^\circ$ .

$[\frac{y}{\theta}]_i$	$\frac{N_f}{N_i}$	$\frac{v_f}{v_i}$	$\frac{f}{f_i}$	Lattice periods
0.2	1	1	1.22(1.21)	30
0.2	1	2	1.20	30
0.2	4	2	1.28(1.26)	30
0.1	1	1	1.84(1.69)	15
0.1	1	2	1.70	15
0.1	4	2	1.90(2.00)	15

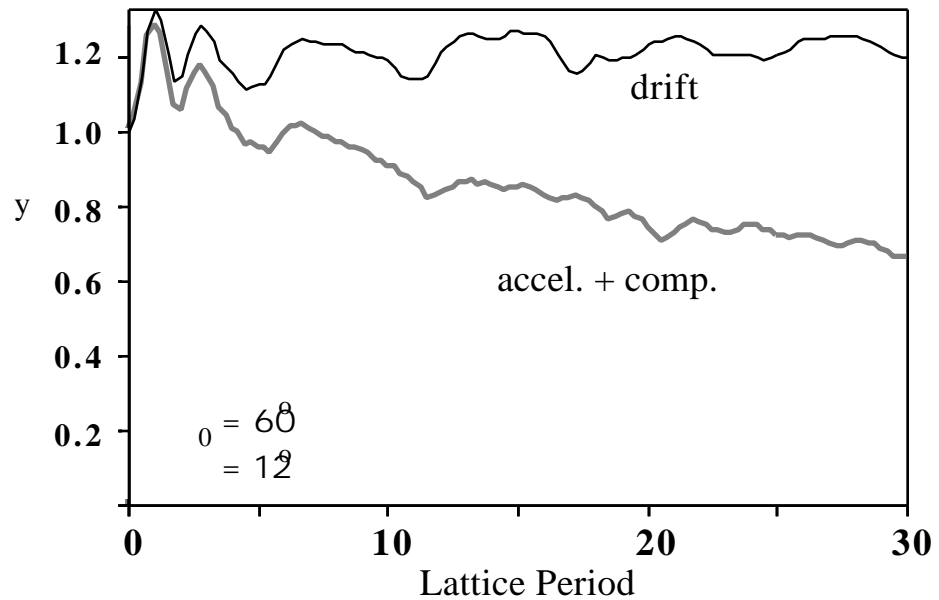


Figure 1. Time history of emittance.

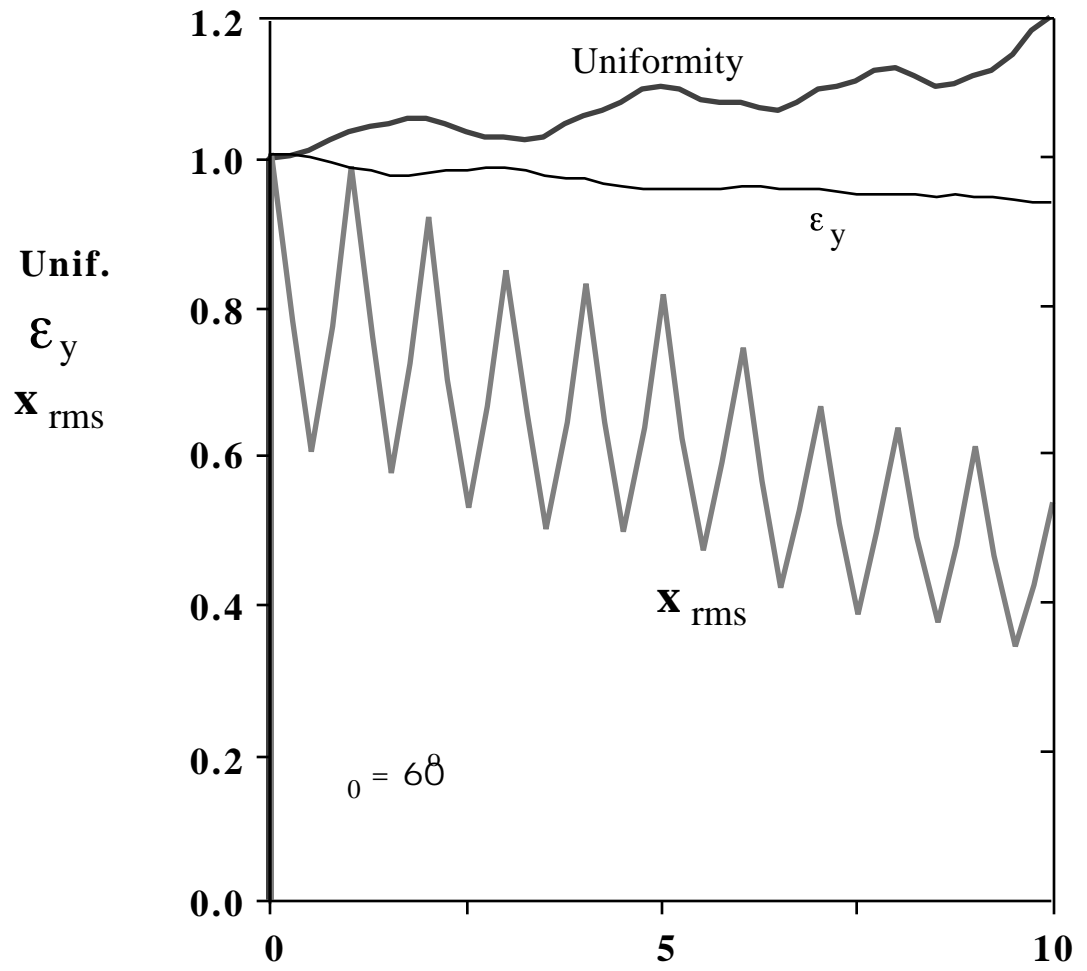


Figure 2. Evolution of  $x_{rms}$ ,  $\epsilon_y$  and uniformity. The uniformity is defined by  $\frac{\langle x^4 \rangle}{2 \langle x^2 \rangle^2}$  which is unity for a uniform density distribution, and greater or less than unity for peaked and hollow beams, respectively.